Hold on to your homework!

4.38 A catapult is designed to shoot water balloons at a speed of 35 miles per hour (mph). Suppose that the catapult is aimed so that there is a constant 10 mph wind blowing in the same direction as the balloon. This wind will add that speed to the horizontal velocity of the balloon.

(a) What is the optimal angle θ from the horizontal that will allow the balloon to travel the greatest distance?

(b) If that optimal angle is used, what horizontal distance does the balloon travel?

$$\Rightarrow \begin{pmatrix}\text{horiz}\\ \text{velocity}\end{pmatrix} = 51.33\cos\theta + 14.67\,\text{ft/s}$$

$$\begin{matrix}\text{distance}\\ \text{travelled}\end{matrix} = F(\theta) = \begin{pmatrix}\text{horiz}\\ \text{velocity}\end{pmatrix}\begin{pmatrix}\text{time at}\\ \text{end}\end{pmatrix} = (51.33\cos\theta + 14.67)\,T_{\text{bottom}}.$$

vertical velocity $V(t) = 51.33\sin\theta - 32t = 0$ at high pt.

$$T_{\text{bottom}} = 2\cdot(\text{Time to high pt.})$$

Time to high pt $= t$

$$51.33\sin\theta - 32t = 0$$

$$\Rightarrow t = \frac{51.33\sin\theta}{32}$$

$$T_{\text{bottom}} = 2t = \frac{51.33\sin\theta}{16}$$

$$\Rightarrow F(\theta) = \left(51.33\cos\theta + 14.67\right)\left(\frac{51.33\sin\theta}{16}\right)$$

horiz velocity. time to ground

$$0 \le \theta \le \frac{\pi}{2}$$

maximize

$$F'(\theta) = \left(-51.33\sin\theta\right)\left(\frac{51.33\sin\theta}{16}\right) + \left(51.33\cos\theta + 14.67\right)\left(\frac{51.33\cos\theta}{16}\right)$$

$$= -164.67\sin^2\theta + 164.67\cos^2\theta + 47.06\cos\theta = 0$$

$$= -164.67\left(1-\cos^2\theta\right) + 164.67\cos^2\theta + 47.06\cos\theta = 0$$

$$= 329.34\cos^2\theta + 47.06\cos\theta - 164.67 = 0$$

$$\Rightarrow \cos\theta = \frac{-47.06 + \sqrt{47.06^2 + 4\cdot 329.34\cdot 164.67}}{2(329.34)}$$

$$\Rightarrow \cos\theta = 0.639$$

$$\Rightarrow \theta = \arccos(0.639) = \boxed{\begin{array}{c}0.877\\ 50.26^\circ\end{array}}$$

$F'(\theta)$ + + + + 0 − − − −

0   0.877 $\frac{\pi}{2}$ θ

$F(\theta)$  $\Rightarrow \theta = 0.877$ is global max!

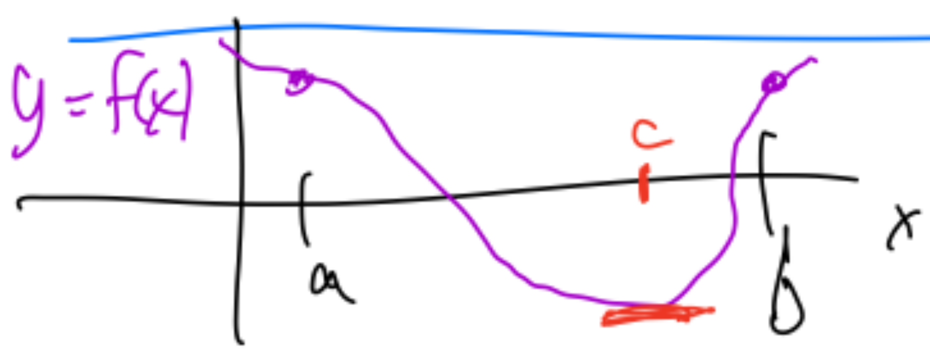
(b) How far did the balloon go?

$\boxed{F(0.877) = 117 \text{ ft.}}$

Background theory about derivatives.

<u>Rolle's Theorem</u>

If f is a function that is differentiable on $[a, b]$, and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.



Sketch of proof: If f is constant, then every $c \in (a,b)$ has $f'(c) = 0$.
Otherwise, there is some pt x where $f(x) \neq f(a)$
$\Rightarrow \exists$ absolute max or abs. min that $f(b)$
is an interior pt.

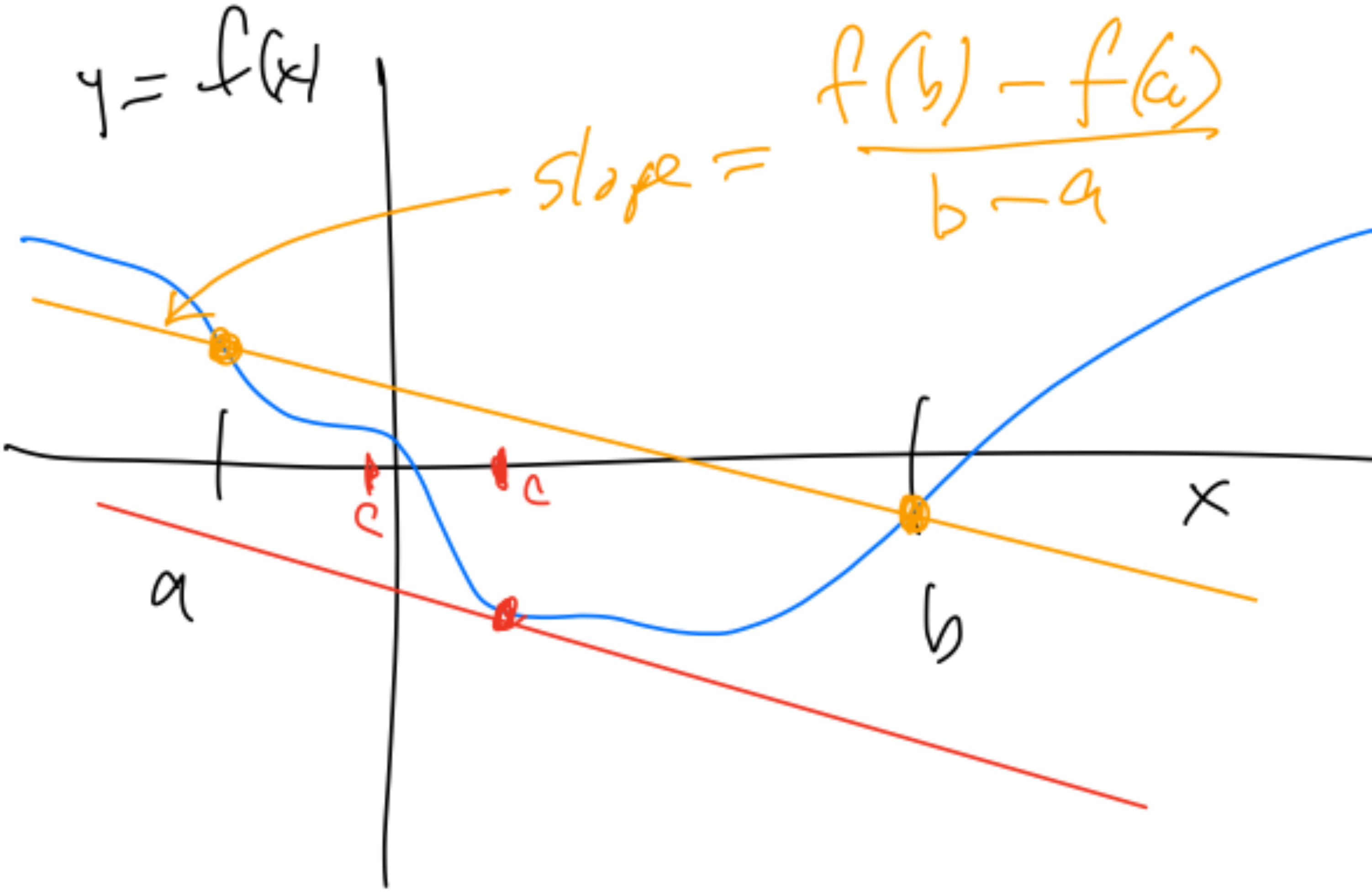
$\exists$ = "there exists".

$\square$

Mean Value Theorem.

If f is a differentiable function on $[a,b]$, then $\exists\, c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Way to prove this:

let $g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)x$

You can check that

$g(a) = g(b) \Rightarrow g(x)$

satisfies Rolles' Theorem

$\Rightarrow \exists\, c$ where $g'(c) = 0$

$g'(c) = f'(c) - \left(\frac{f(b)-f(a)}{b-a}\right) = 0$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}. \quad \boxed{}$$

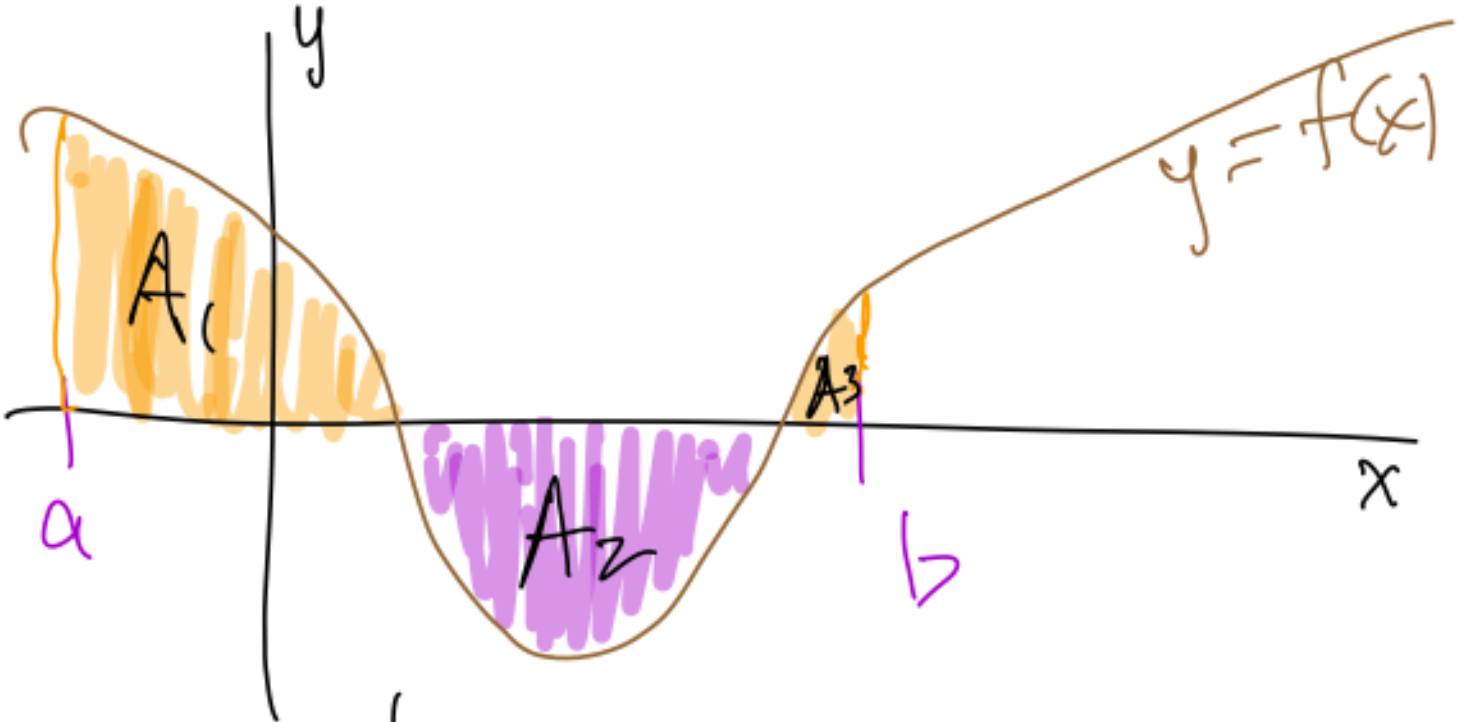
New topic – Integrals:

Given a function $f(x)$ on an interval $[a,b]$,

$$\int_a^b f(x)\,dx =$$

(intuitive definition)

= (signed) area under $y = f(x)$ between $x=a$ & $x=b$.

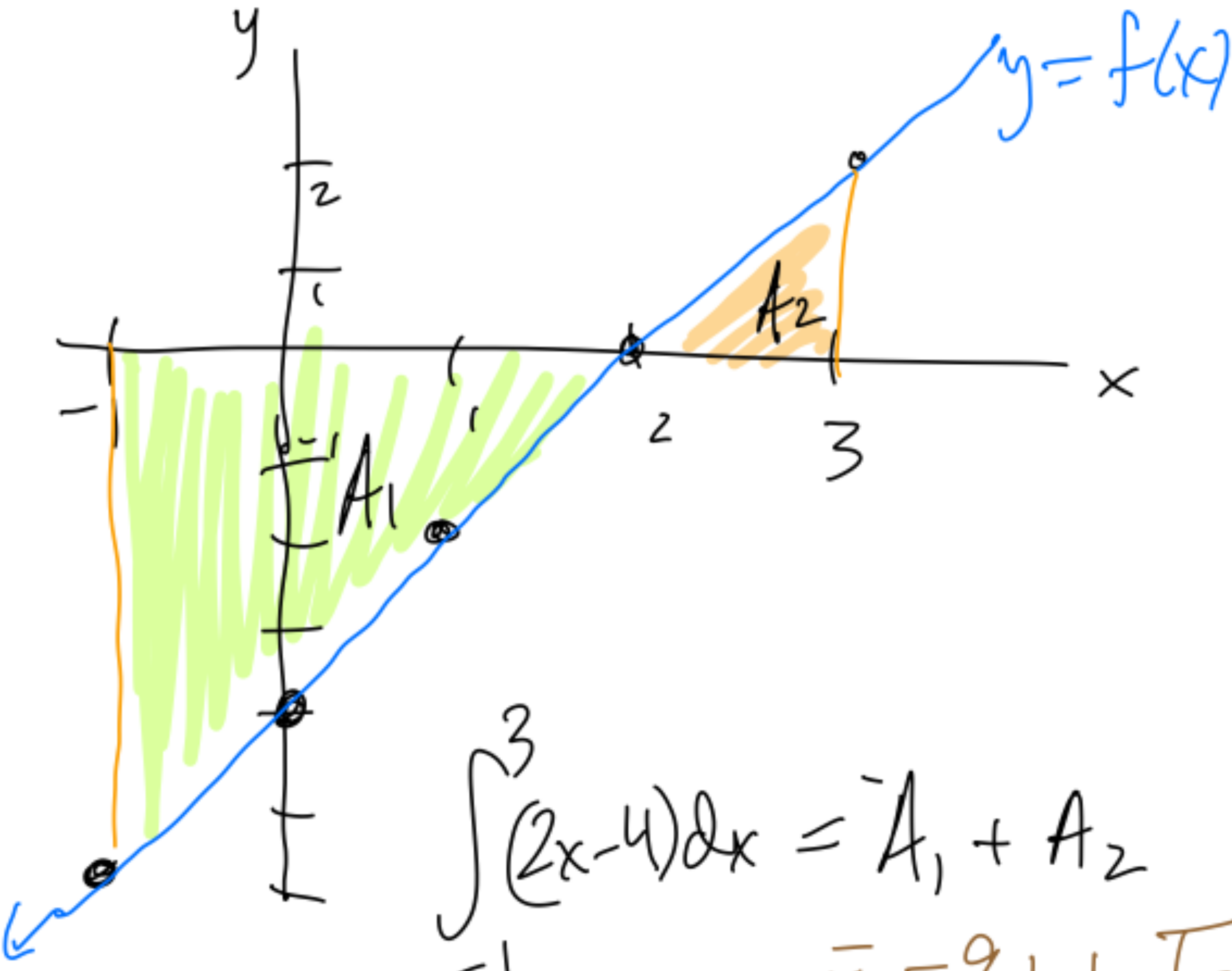


$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

in this picture.

__Examples:__

Calculate $\int_{-1}^{3} (2x-4)\,dx$



$$\int_{-1}^{3}(2x-4)\,dx = -A_1 + A_2$$

$$= -9 + 1 = \boxed{-8}$$

$$A_1 = \tfrac{1}{2}bh = \tfrac{1}{2}(3)(6) = 9$$

$$A_2 = \tfrac{1}{2}bh = \tfrac{1}{2}(1)(2) = 1$$

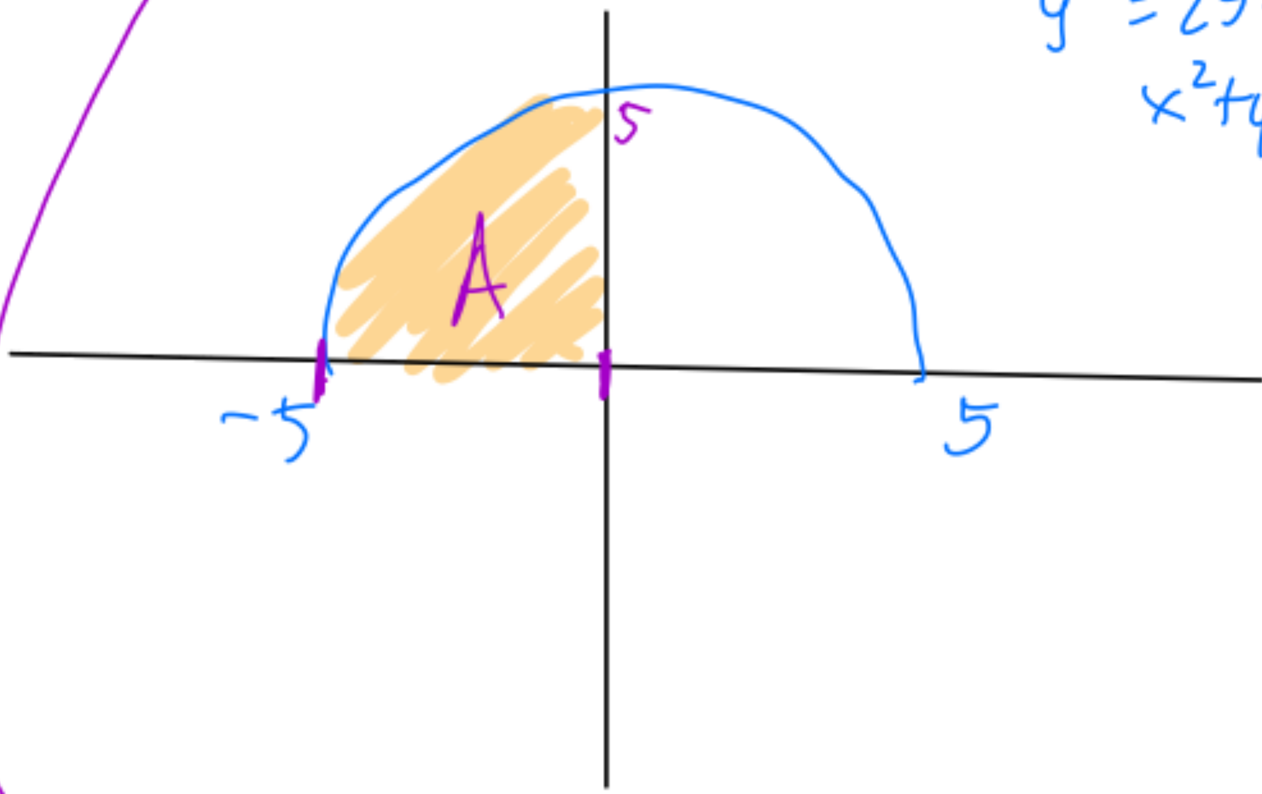
Example

$$\int_{-5}^{0} \sqrt{25-x^2}\,dx$$

$y = \sqrt{25-x^2}$

$y^2 = 25 - x^2$

$x^2 + y^2 = 25$



$= A = \frac{1}{4}$ (area of disk of radius 5)

$= \frac{1}{4}\pi(5)^2 = \boxed{\frac{25\pi}{4}}$

True Definition of the Definite Integral $\int_a^b f(x)\,dx$

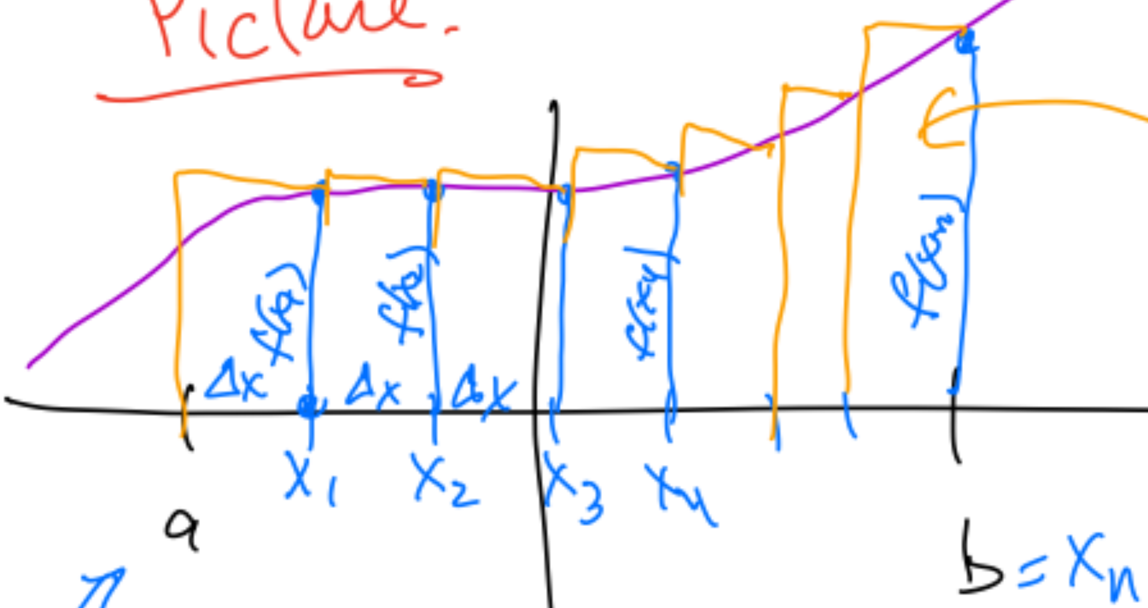
summation

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot \Delta x,$$

where $\Delta x = \dfrac{b-a}{n}$ & $x_k = a + k\Delta x$.

Picture:

$y = f(x)$



sum of rectangle areas $= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \dots f(x_n)\Delta x.$

subdividing $[a,b]$ into n pieces

$$= \sum_{k=1}^{n} f(x_k)\Delta x.$$

Summation notation:

Examples

- $$\sum_{k=1}^{5} 2k = \underset{k=1}{2k} + \underset{k=2}{2k} + \underset{k=3}{2k} + \underset{k=4}{2k} + \underset{k=5}{2k}$$

$$= 2 + 4 + 6 + 8 + 10 = 30$$

- $$\sum_{k=0}^{4} 3\cdot 2^k = \underset{k=0}{3\cdot 2^k} + \underset{k=1}{3\cdot 2^k} + \underset{k=2}{3\cdot 2^k} +$$

$$+ \underset{k=3}{3\cdot 2^k} + \underset{k=4}{3\cdot 2^k}$$

$$= 3 + 6 + 12 + 24 + 48$$

$$= \boxed{93}.$$

- $\frac{1}{n} \cdot \sum_{k=1}^{n} (3k-1)$

$$= \frac{1}{n} \cdot \Big(\underbrace{3k-1}_{k=1} + \underbrace{3k-1}_{k=2} + \underbrace{3k-1}_{k=3} + \cdots + \underbrace{3k-1}_{k=n} \Big)$$

$$= \frac{1}{n} \cdot \left(2 + 5 + 8 + 11 + \cdots + 3n-1 \right)$$

- $\sum_{k=1}^{n} 1 = \underbrace{1}_{k=1} + \underbrace{1}_{k=2} + \underbrace{1}_{k=3} + \cdots + \underbrace{1}_{k=n}$

$$= \boxed{n}.$$

Quiz

1. What is your NAME?

2. Are you an AI?

3. Simplify

   a. $$\frac{x^3\cos(x)}{x^4+x^6}$$

   b. $$\left(\cos(3x)\right)'$$

   c. $$\left(\arctan(x^3)\right)'$$

   d. $$\left(x \cdot e^{2x}\right)'$$